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Waves in a hot plasma

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In studying wave propagation in a hot plasma, we treat the dynamics of the medium by kinetic theory rather than by continuum mechanics. The theory thus combines Maxwell's equations with a transport equation in phase space (the Vlasov equation). An outline of the required procedure will be given.

Some of the results are in close agreement with those of the fluid treatment provided the specific heat ratio is appropriately chosen. This is generally the case if the phase speed of the waves well exceeds the thermal speed of the electrons and, for a magnetized plasma, the frequency is not close to a harmonic of the cyclotron frequency.

New phenomena are found if there are particles whose unperturbed motion is in resonance with the wave field. In the unmagnetized case this results in Landau damping or in instabilities, the latter being analogous to the mechanism of the laser. In the magnetized case there are, in addition, completely new modes of propagation for waves travelling approximately normal to the applied field.

Many of these phenomena find direct application in ionospheric phenomena and diagnostics.

1. Introduction

In this paper we shall consider the extensions necessary to the theory, described in the previous paper, if it is desired to account for thermal motion when treating wave propagation in a plasma. This problem attracted rather little attention in the 1930s (when the cold plasma theory was being intensively developed) and the modern investigations in the field can be dated rather accurately as starting from the famous paper by Landau in 1946. Most of the work to be reviewed here was complete by 1960, but I shall draw attention to some quite recent and valuable computations for the dispersion relation. Applications of the theory in several geophysical contexts will be briefly indicated by way of introduction to the detailed study made by later speakers at this meeting.

2. The Vlasov equation

When we use kinetic theory to describe the motion of gas we work with the Boltzmann function $f(\boldsymbol{x}, \boldsymbol{v}, t)$ defined as the density of particles in phase space at position $\boldsymbol{x}$, with velocity $\boldsymbol{v}$ at time t. This satisfies the Boltzmann equation, which merely expresses the conservation of particles

$$\frac{\partial f}{\partial t}+\boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}}+\boldsymbol{a}\cdot\frac{\partial f}{\partial \boldsymbol{v}}=\left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}}. \tag{2.1}$$

Here $\boldsymbol{a}(\boldsymbol{x}, \boldsymbol{v}, t)$ is the smooth part of the acceleration of a particle at $\boldsymbol{x}$ with velocity $\boldsymbol{v}$. By 'smooth part' we mean that small scale contributions to fields have been averaged out and transferred to the term $(\partial f/\partial t)_{\mathrm{c}}$ on the right, known as the 'collision term'; to be more precise about this would involve a lengthy digression.

In a hot plasma we shall want to describe one or more species of charged particles, each with a Boltzmann function satisfying (2.1). The charge and current densities are

$$\rho(\boldsymbol{x},t) = \sum_s e_s \int f_s \,\mathrm{d}^3 v, \quad \boldsymbol{j}(\boldsymbol{x},t) = \sum_s e_s \int \boldsymbol{v}_s f_s \,\mathrm{d}^3 v, \tag{2.2}$$

where s labels the species and e_s is the charge on a particle of species s. We may take, in obvious notation

$$\boldsymbol{a}_s = (e_s/m_s)\,(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})+\boldsymbol{g}. \tag{2.3}$$

Here the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ may be partly external (e.g. the geomagnetic field) and partly generated within the plasma itself. To include the latter we write down Maxwell's equations, inserting ρ and $\boldsymbol{j}$ from (2.2). This automatically results in a 'smoothed' $\boldsymbol{a}$ since the use of $f(\boldsymbol{x}, \boldsymbol{v}, t)$ itself, f being differentiable, means that we cannot perceive the individual particles.

The system of equations remains incomplete unless we can formulate $(\partial f/\partial t)_c$. 'The Vlasov approximation', which will be adopted throughout what follows, consists of neglecting this term. This means we neglect the microscopic electric fields implicit in the individuality of the particles, as well as collisions with neutral particles. For many wave phenomena, both laboratory and natural, this is believed to be adequate.

Mathematically, we have to solve the collisionless Boltzmann equation for each species

$$\frac{\partial f_s}{\partial t}+\boldsymbol{v}\cdot\frac{\partial f_s}{\partial \boldsymbol{x}}+\boldsymbol{a}_s\cdot\frac{\partial f_s}{\partial \boldsymbol{v}} = 0, \tag{2.4}$$

together with Maxwell's equations; these equations are coupled by (2.2) and (2.3). The result is a set of non-linear integro-differential equations. From now on we will omit the gravitational acceleration $\boldsymbol{g}$ included for completeness in (2.3).

3. Waves in an unmagnetized plasma: Landau's theory

3.1. *Linearization*

To investigate wave propagation in a hot plasma we can proceed as in magnetoionic theory by considering an unperturbed state (suffix $_0$) and small perturbations (suffix $_1$). The unperturbed variables themselves form a solution of the governing equations, and vary only slowly, if at all, in space and time. The perturbations represent waves of low amplitude, and, in the elementary parts of the subject, we linearize the equations by neglecting higher powers than the first. Thus we write $f = f_0 + f_1$ (for each species), $\boldsymbol{E} = \boldsymbol{E}_0+\boldsymbol{E}_1$, $\boldsymbol{B} = \boldsymbol{B}_0+\boldsymbol{B}_1$. The resulting set of equations describing the waves are linear integro-differential equations. Naturally the simplest case is obtained by taking the unperturbed solution to be uniform in space and time, and the present review is restricted to that case. We shall also assume that $\rho_0 = \boldsymbol{j}_0 = 0$ and $\boldsymbol{E}_0 = 0$ in the absence of the waves, so the species simply combine to form a neutral current-free plasma. We then have $f_0 = f_0(\boldsymbol{v})$ only. The zero and first order equations resulting from (2.4) are

$$(\boldsymbol{v}\times\boldsymbol{B}_0)\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} = 0, \tag{3.1}$$

$$\frac{\partial f_1}{\partial t}+\boldsymbol{v}\cdot\frac{\partial f_1}{\partial \boldsymbol{x}}+\frac{e}{m}(\boldsymbol{v}\times\boldsymbol{B}_0)\cdot\frac{\partial f_1}{\partial \boldsymbol{v}}+\frac{e}{m}(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}_1)\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} = 0 \tag{3.2}$$

($\boldsymbol{E}_1$ may now be designated as $\boldsymbol{E}$ since $\boldsymbol{E}_0 = 0$). We seek solutions in which all wave quantities are harmonic in space and time

$$f_1, \boldsymbol{E}, \boldsymbol{B}_1 \propto \exp[\mathrm{i}(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})]. \tag{3.3}$$

In magnetoionic theory this procedure converts differential equations to homogeneous algebraic equations, and gives the dispersion relation as the consistency condition. For the hot plasma we still have differentiations with respect to $\boldsymbol{v}$ in (3.2) and integrations over $\boldsymbol{v}$ in forming ρ and $\boldsymbol{j}$, so that forming the dispersion relation, if any exists, will involve solving for the behaviour of f_1 in velocity space, then computing integrals. It is at this point that the two treatments diverge.

A further contrast can be made between the unmagnetized ($\boldsymbol{B}_0 = 0$) case, to which this section will be devoted, and the magnetized case ($\boldsymbol{B}_0 \neq 0$). If $\boldsymbol{B}_0 = 0$, (3.1) imposes no condition on $f_0(\boldsymbol{v})$, so it can be any distribution (subject of course to the requirements $f_0 \geqslant 0$ and that integrals over $\boldsymbol{v}$ needed to construct the particle density, etc., be convergent). Moreover, $\boldsymbol{B}_0 = 0$ removes the third term from (3.2), so there is not the need to solve a differential equation for f_1. By the same token, restoring $\boldsymbol{B}_0 \neq 0$ to the equations alters their character since any $\boldsymbol{B}_0$, however small, places a new condition on f_0 and increases the order (from 0 to 1) of the differential equation (3.2); introducing $\boldsymbol{B}_0$ is therefore non-trivial, and attempts to check the calculations by letting $\boldsymbol{B}_0 \to 0$ are often confusing.

3.2. *Solution of equations: causality and the Landau prescription*

With the above assumptions (3.2) may be written

$$\mathrm{i}(\omega - \boldsymbol{k}\cdot\boldsymbol{v})f_1 = -(e/m)\,(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}_1)\cdot\partial f_0/\partial\boldsymbol{v}, \tag{3.4}$$

where $f_1(\boldsymbol{v})$ is now the complex representative of the plane wave perturbations. 'Solving' this equation requires only the division by the factor $\omega - \boldsymbol{k}\cdot\boldsymbol{v}$, and we may substitute f_1 into the integrals

$$\rho_1 = \int f_1\,\mathrm{d}^3v, \quad \boldsymbol{j}_1 = \int \boldsymbol{v}f_1\,\mathrm{d}^3v \tag{3.5}$$

(with summation over species). On combining with Maxwell's equations

$$\left.\begin{aligned} \boldsymbol{k}\times\boldsymbol{E} &= \omega\boldsymbol{B}_1, \\ -\mathrm{i}\mu_0^{-1}\boldsymbol{k}\times\boldsymbol{B}_1 &= \boldsymbol{j}_1 + \epsilon_0\mathrm{i}\omega\boldsymbol{E}_1, \end{aligned}\right\} \tag{3.6}$$

we have a closed homogeneous system for the ratios $f_1 : \boldsymbol{E} : \boldsymbol{B}_1$, with a consistency condition which, it is hoped, will play the role of the dispersion relation. This condition has ω and $\boldsymbol{k}$ implicitly involved in integrals over velocity space; the integrands contain $\partial f_0/\partial\boldsymbol{v}$, and all have the factor $(\omega - \boldsymbol{k}\cdot\boldsymbol{v})^{-1}$; by contrast in magnetoionic theory the corresponding equations involve ω and $\boldsymbol{k}$ in polynomials.

Clearly the handling of the dispersion relation is more laborious; but a difficulty of principle beset the earlier investigators in this field. As there are no dissipative mechanisms in the physical system envisaged, we must expect solutions with ω and $\boldsymbol{k}$ purely real. But in that case how do we deal with the singularity in the integrals that results from the vanishing of $\omega - \boldsymbol{k}\cdot\boldsymbol{v}$ for certain values of $\boldsymbol{v}$?

An early suggestion by Vlasov was that one should take the Cauchy principal part of the integral in such cases. But it can be shown that, in a substantial class of circumstances, this leads to a dispersion relation with no solution at all.

In his pioneering paper of 1946, Landau considered an initial-value problem in which f_1 is proportional to $\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})$ at $t = 0$, without as yet assuming a time dependence $\exp(\mathrm{i}\omega t)$. He solved this problem, formally, by Laplace transformation, and the Bromwich inversion formula provides the answer. An examination of the expression for the electric field, $\boldsymbol{E}$, shows that, after a sufficient time, the major contribution is indeed expressible in the form $\boldsymbol{E}\exp(\mathrm{i}\omega t)$, where ω satisfies the dispersion relation, but with the important difference that there is a new prescription for dealing with the singular integral.

The result may be described briefly by saying that when solving (3.4) one must start with ω well below the real axis; such would correspond to the problem of calculating f_1, as the response to an exponentially *growing* $\boldsymbol{E}$, that is a field switched on in the past. The integrals (3.5) are correspondingly interpreted, and there is no difficulty since $\omega - \boldsymbol{k}\cdot\boldsymbol{v}$ does not vanish for any real $\boldsymbol{v}$. When we need to extend the definition of these integrals to the case ω real, or ω in the upper half plane, an analytic continuation with respect to ω is necessary. The integration over the component of $\boldsymbol{v}$ parallel to $\boldsymbol{k}$ then proceeds along an indented contour instead of being along the real axis. This contour is shown in figure 1, taking $\boldsymbol{k} = (k, 0, 0)$, $k > 0$. In the resulting dispersion equation, it commonly happens that the solution $\omega(\boldsymbol{k})$, for real $\boldsymbol{k}$, is complex and indicates damping, known as 'Landau damping'.

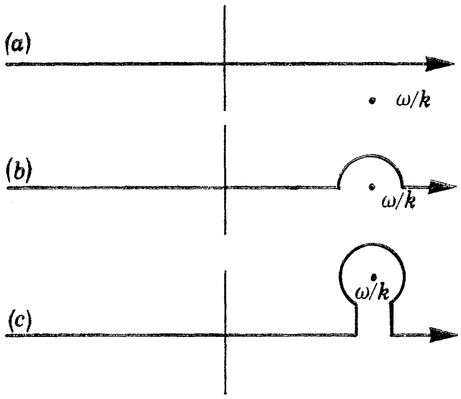

FIGURE 1. The Landau contour.

This theory was received with some scepticism. From a somewhat abstruse mathematical derivation there appeared to emerge a dissipative mechanism although none was present in the original physics; irreversibility had arisen although the original differential equations were demonstrably time-reversible. Moreover, in 1946 there seemed little prospect of any experimental check on the work.

Space does not allow a detailed discussion of the point here, but we note that the theory has been accepted. Mathematically, the point is that the set of integro-differential equations alone do not form a completely posed problem, even for the consideration of plane waves. The class of initial or boundary values for which the equations are to be solved must also be taken into account (as was done automatically by Landau's use of the Laplace transformation). This is in close analogy with radiation theory, where Maxwell's equations alone cannot distinguish between retarded and advanced potentials. Physically, the essence of Landau damping is that it is due to the existence of particles whose unperturbed motion puts them in resonance with the wave field; this is just what happens if $\omega = \boldsymbol{k}\cdot\boldsymbol{v}$ and ω, $\boldsymbol{k}$ are real. If ω is very nearly real, conditions are still close to resonance, and the result is a strong coupling between the wave and that particular group of particles. This coupling results in a transfer of energy between the two, in a sense that depends on the sign of $\partial f_0/\partial \boldsymbol{v}$; in the one case we have Landau damping, in the other instability in which the energy of the wave is abstracted from the kinetic energy of the particles. This latter phenomenon is very similar to the action of a laser.

Finally, a number of experimental investigations have tended to confirm the Landau theory.

3.3. *Plasma oscillations*

Oscillations in a cold unmagnetized plasma separate into plasma oscillations, which are purely longitudinal, and electromagnetic waves which are purely transverse. If f_0 is reasonably close to isotropic, the same is true in the hot plasma. Let us consider plasma oscillations, in which the electrons oscillate in the direction of propagation; we can regard the ions as at rest.

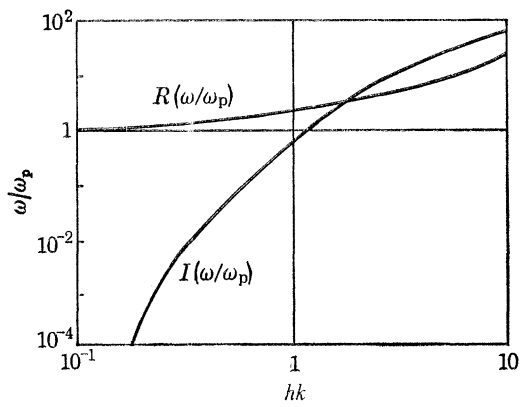

FIGURE 2. The dispersion relation for a Maxwellian plasma.

The results for simple cases are well typified by the maxwellian distribution (density N, temperature T)

$$f_0 = N(m/2\pi KT)^{\frac{3}{2}} \exp[-mv^2/2KT]. \tag{3.7}$$

Taking $\boldsymbol{k} = (k, 0, 0)$, $(k > 0)$ the velocity distribution in the plane perpendicular to $\boldsymbol{k}$ is irrelevant and can be integrated out once and for all. Maxwell's equations may be replaced by Poisson's equation, and the dispersion relation resulting for carying out the procedure outlined above is

$$zG(z) = 1 + h^2k^2, \tag{3.8}$$

where

$$z = \frac{\omega/\omega_p}{2^{\frac{1}{2}}hk}, \quad \omega_p = \left(\frac{Ne^2}{\epsilon_0 m}\right)^{\frac{1}{2}}, \quad h = \left(\frac{\epsilon_0 KT}{Ne^2}\right)^{\frac{1}{2}}$$

(ω_p is the plasma frequency, h the Debye length) and $G(z)$ is the error function of imaginary argument

$$G(z) = 2\exp(-z^2)\int_{-\mathrm{i}\infty}^{z} \exp(p^2)\,\mathrm{d}p. \tag{3.9}$$

Figure 2 shows the solution ω/ω_p plotted against hk (assumed real). For long waves ($hk \ll 1$), we have to very good approximation

$$\omega = \omega_p(1 + \tfrac{3}{2}h^2k^2) \tag{3.10}$$

and Landau damping is negligible. Equation (3.10) is just what one obtains by modifying the simple cold theory by adding to the electrostatic restoring force a mechanical one due to the pressure of the electrons, with the proviso that the ratio of specific heats be taken to be 3. The reason why Landau damping is so slight is that in this case the phase speed ($\simeq \omega_p/k$) well exceeds the sound speed ($\omega_p h$), so there are very few electrons in resonance with the wave. For short waves, Landau damping becomes very severe, and in practice such waves are unobservable and of no importance.

3.4. *Ion waves*

In a cold plasma, inclusion of positive ions makes only a minute correction to the treatment of plasma oscillations, since $m_i \gg m_e$; the same is true in a hot plasma. If we add a simple pressure term, as described earlier, the modified cold plasma theory suggests the existence of a mode of propagation which could properly be called a sound wave. The ions and electrons would move together, with very little charge separation, the speed being given by the usual formula (inserting the *total* pressure and mass density). This is however a useful warning against accepting such a simple idea without criticism and indeed kinetic theory shows that such waves would be heavily Landau damped. This is because the sound speed is only a little higher than the ion thermal speed, so that there are plenty of ions which could be in resonance with such a wave. This wave can be made to escape Landau damping if we can increase its speed without increasing the ions' thermal speed, and this may be achieved by having the electrons considerably hotter than the ions. This point is of some interest in connexion with the theory of incoherent scatter.

3.5. *Transverse waves*

In cold plasma theory, the electromagnetic wave satisfies

$$\omega^2 = \omega_p^2 + c^2k^2.$$

Kinetic theory makes virtually no difference to this result unless we are concerned with relativistic temperatures. Moreover there is no Landau damping since the phase speed exceeds c.

For an *anisotropic* f_0, kinetic theory introduces a quite new mode of propagation which can be unstable (the Weibel–Kahn instability).

3.6. *Stability of non-Maxwellian distributions*

Returning to electrostatic modes, take $\boldsymbol{k}$ in a fixed direction and let $f_0(v_1)$ be the result of integrating $f_0(\boldsymbol{v})$ with respect to the velocity components perpendicular to $\boldsymbol{k}$, v_1 being the component parallel to $\boldsymbol{k}$. For a given k one can ask whether Im (ω) is positive or negative, indicating damping or instability, and this can be determined by actual calculation of ω or by using complex variable techniques for locating zeros of a function, also known as Nyquist diagrams. The same idea has also been used in the case of magnetized plasmas to deal with the electrojet instability. In the unmagnetized case, the technique yields an answer to the more general question whether there is an instability for *any* k (but is still restricted to the direction selected). Unstable distributions are characterized by having at least two maxima in $f_0(v_1)$. The existence of the minimum that must occur between these is a necessary but not sufficient condition for instability. The complete necessary and sufficient condition was given by O. Penrose (1960); the additional part amounts to requiring that the minimum in $f_0(v_1)$ be sufficiently deep, in a sense there made precise.

4. Waves in a magnetized plasma

4.1. *Basic equations*

We consider next a plasma whose unperturbed state is uniform in space and time, and with uniform unperturbed magnetic field $\boldsymbol{B}_0$. Then f_0 is restricted by (3.1) to take the form

$$f_0 = f_0(v_\perp, v_\parallel) \quad \text{only,} \tag{4.1}$$

where $v_\parallel$, $v_\perp$ are the velocity components parallel and perpendicular to the field. We use a similar notation $k_\parallel$, $k_\perp$ for $\boldsymbol{k}$. Equation (4.1) prompts the use of cylindrical coordinates for velocity space, say $(v_\perp, \phi, v_\parallel)$. Turning to (3.2), we proceed as before and assume the exponential form (3.3) with ω (in the first instance) in the lower half plane. Then (with alternative signs for ions and electrons; $\Omega = eB_0/m$),

$$\mathrm{i}(\omega - \boldsymbol{k}\cdot\boldsymbol{v})f_1 \pm \Omega\, \partial f_1/\partial\phi = -(e/m)\,(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}_1)\cdot\partial f_0/\partial\boldsymbol{v}. \tag{4.2}$$

Since this contains derivatives only with respect to ϕ, and not $v_\perp$ or $v_\parallel$, we have to deal only with an ordinary differential equation. This is readily solved and the result used to compute ρ and $\boldsymbol{j}$ by (3.5). Combining with Maxwell's equations one obtains, formally, a dispersion relation. We may then allow ω to become real or migrate into the upper half plane, continuing analytically.

This relation is in general in the form of a 3×3 determinant, of which every entry involves integrals over $v_\perp$ and $v_\parallel$, to be performed when f_0 is specified. A characteristic feature (originating in the use of Cartesian coordinates in physical space and cylindrical coordinates in velocity space) is the profusion of Bessel functions. Another interesting general point is that the resonance condition $\omega - \boldsymbol{k}\cdot\boldsymbol{v} = 0$ noticed before is replaced by $\omega - k_\parallel v_\parallel - n\Omega = 0$, $n = 0, \pm 1, \pm 2, \ldots$.

It would be inappropriate to develop details here owing to the excessive length; the matter has been extensively treated in a number of texts and review articles (Stix 1962; Montgomery & Tidman 1964; Clemmow & Dougherty 1939; Baldwin, Bernstein & Weenink 1969). In the following sections some of the main features of the results are summarized.

4.2. *Bernstein modes*

A commonly used approximation, which much simplifies the analysis, is that of replacing the full set of Maxwell's equations by the equations of electrostatics. Generally, this 'electrostatic approximation' is valid if the phase speed of the waves, as so discovered, is much less than c; but the approximation also overlooks some modes of propagation altogether (even though the speed for such a mode, as correctly treated, is much less than c).

Let us now consider the case of a Maxwellian plasma, so that (3.7) holds; for the present regard the ions as stationary. The dispersion relation is

$$1 + h^2k^2 = z_0\,\mathrm{e}^{-\lambda} \sum_{n=-\infty}^{\infty} I_n(\lambda)\, G(z_n), \tag{4.3}$$

where h, G have the same meaning as before,

$$\lambda = \frac{KTk_\perp^2}{m\Omega^2}, \quad z_n = \frac{\omega - n\Omega}{k_\parallel}\left(\frac{m}{2KT}\right)^{\frac{1}{2}}, \quad n = 0, \pm 1, \ldots \tag{4.4}$$

(so z_0 is what was called z in the field free case). I_n is defined by $I_n(\lambda) = \mathrm{i}^{-n}J_n(\mathrm{i}\lambda)$, the modified Bessel function.

Waves travelling parallel to $\boldsymbol{B}_0$ are unaffected by the field, and for $\boldsymbol{k}$ at a modest angle to $\boldsymbol{B}_0$ no new effects of practical interest arise. But if we let $\boldsymbol{k}$ approach the position of being perpendicular to $\boldsymbol{B}_0$, new modes of propagation are found which can exist for any wavelength. When $k_\parallel = 0$ Landau damping disappears altogether and there is an infinite collection of propagating modes. The physical reason for the absence of damping is that if ω is not a harmonic of the gyrofrequency (even though it may be close to one) there are now no particles in resonance with the wave. These undamped modes are often called 'Bernstein modes', after the investigation by Bernstein (1958). The dispersion relation (4.3) becomes in this case

$$\frac{\omega_p^2}{\Omega\lambda} e^{-\lambda} \sum_{n=-\infty}^{\infty} \frac{nI_n(\lambda)}{\omega - n\Omega} = 1. \tag{4.5}$$

It is easy to show that for *small* $k_\perp$ (and therefore λ), there is a root closed to the hybrid frequency $\omega_H = (\omega_p^2 + \Omega^2)^{\frac{1}{2}}$ and a root close to every harmonic $n\Omega$ *except* $n = 1$. Likewise, for *large* $k_\perp$, there is a root close to every harmonic, including the first (but none close to hybrid frequency). Numerical treatment shows how these branches are connected. The detailed forms of the curves depend on ω_p/Ω, but we show an example in figure 3. It will be noticed that for $\Omega \leqslant \omega \leqslant \omega_H$ propagation at any frequency is possible, while for $\omega > \omega_H$ propagation only occurs in narrow bands just above each harmonic.

The Bernstein modes have been investigated in considerable detail in laboratory experiments. They have also figured prominently in the theory of resonances observed in topside sounders, owing to the low values of group velocity associated with these modes.

Inclusion of ion motion leads to a corresponding set of modes near the harmonics of the gyrofrequency. Some properties of these will be mentioned later.

4.3. *More exact treatment*

The results just discussed may be called the 'hot electrostatic' dispersion relation, whereas the magnetoionic theory (including ions if wished) yields a 'cold electromagnetic' dispersion relation. Comparison between the two is not very instructive since each theory neglects something retained by the other; in fact at first sight there appears to be no connexion between them.

To make a connexion, one must make calculations, at least for some sample situations, which make neither approximation. As we have seen the full dispersion relation involves 3×3 determinantal equations. The following special cases provide some simplification, however.

(i) *Propagation parallel to* $\boldsymbol{B}_0$

Here the dispersion relation factorizes, just as in the cold case, into plasma oscillations and two circularly polarized electromagnetic waves; the latter are of course missed altogether by the electrostatic approximation. The new feature here is the Landau damping of these electromagnetic waves, which, as usual, is appreciable if the phase velocity is sufficiently low. The 'whistler mode' often has a phase velocity well below c, but nevertheless (in the geophysical context) well above the thermal speed of the electrons. For a Maxwellian distribution, therefore, there is little effect in practice. A similar remark applied to Alfvén waves and their Landau damping by the ions.

In the presence of a high-energy component in the plasma, depending on the sign of $\partial f_0/\partial \boldsymbol{v}$ at the relevant point, one can have either damping or excitation of the waves. The theory of the mechanism of magnetospheric emissions involves these considerations.

(ii) *Propagation perpendicular to* $\boldsymbol{B}_0$

The dispersion equation again factorizes, into a simple factor which corresponds to the ordinary wave of cold theory and a 2×2 determinantal factor corresponding to the extraordinary wave. There is little of interest in the former since the phase speed in cold theory exceeds c. It is the 'extraordinary' wave which, in the limit $c \rightarrow \infty$ gives the Bernstein modes, as in figure 3. Figure 4 shows the dispersion relation obtained from cold theory, including ion dynamics. Qualitatively, we can expect that figure 3 will need modification in a thin 'wedge' close to the vertical axis, corresponding to phase velocities approaching or exceeding c. It also needs modification if ion dynamics is allowed for, and this will introduce modes close to the origin.

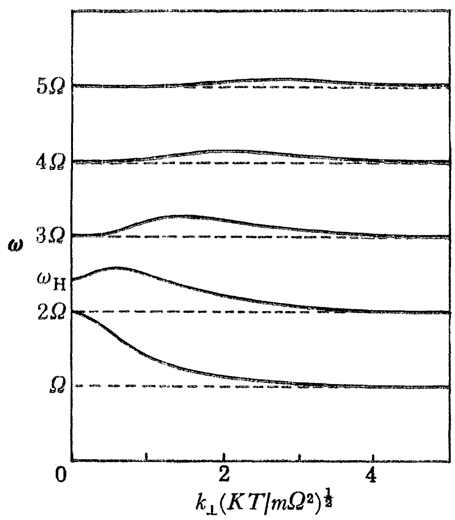

FIGURE 3. Bernstein modes.

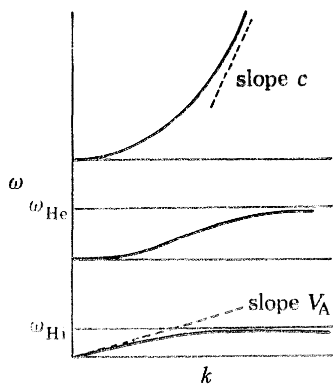

FIGURE 4. Extraordinary wave for a cold plasma.

It is only recently that extensive exact calculations for the complete dispersion relation for these 'generalized Bernstein modes' have been published by Puri, Leuterer & Tutter (1973). A selection of their curves are reproduced here and briefly described. The format differs slightly from that of figures 3 and 4 in that the horizontal scale is logarithmic.

First we confine attention to the electron modes, regarding the ions as immobile. The parameter $\omega_{\mathrm{P}}/\Omega$ is here 4.38, so $\omega_{\mathrm{H}}/\Omega = 4.5$; but we now need a second parameter, for which

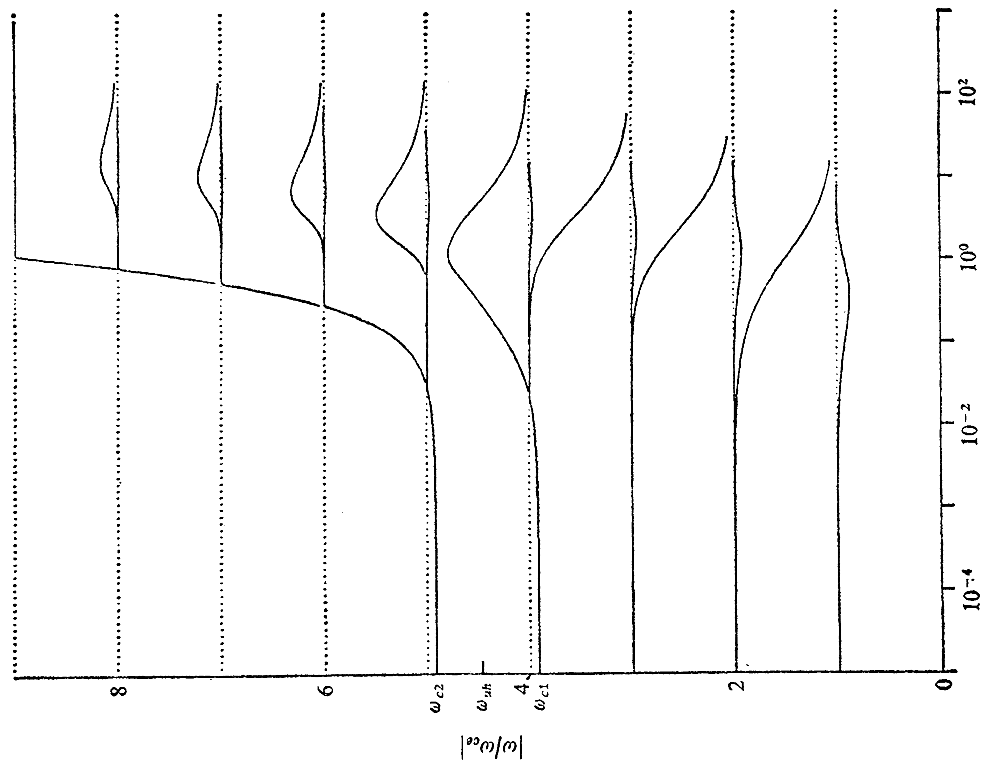

FIGURE 6. Generalized Bernstein modes.

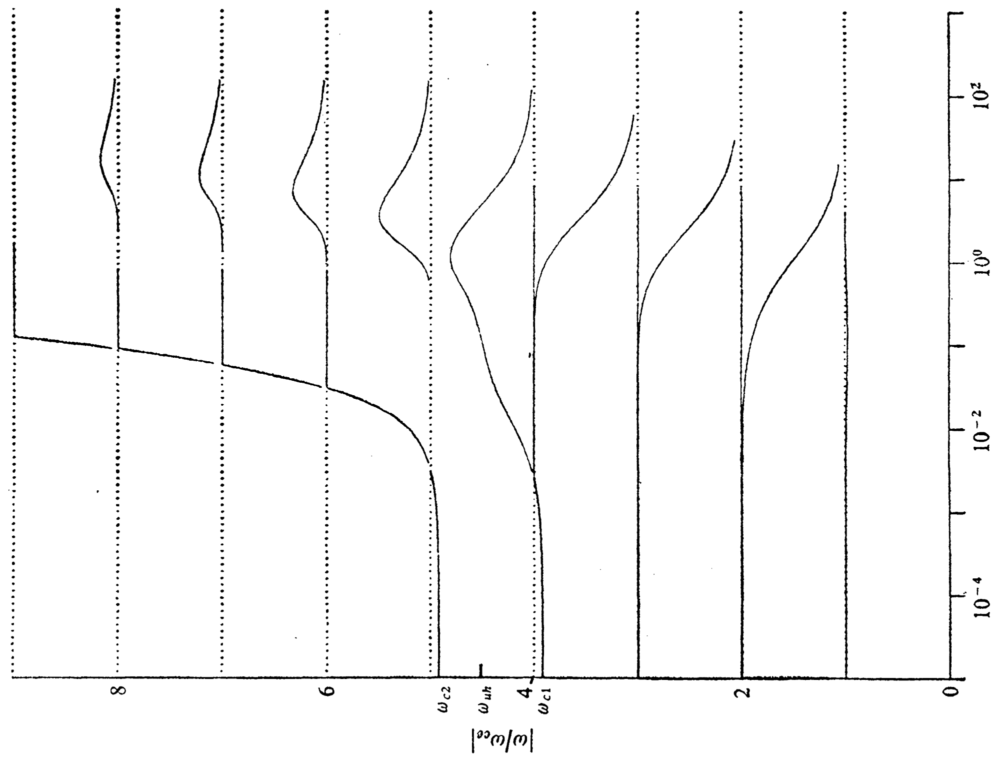

FIGURE 5. Generalized Bernstein modes.

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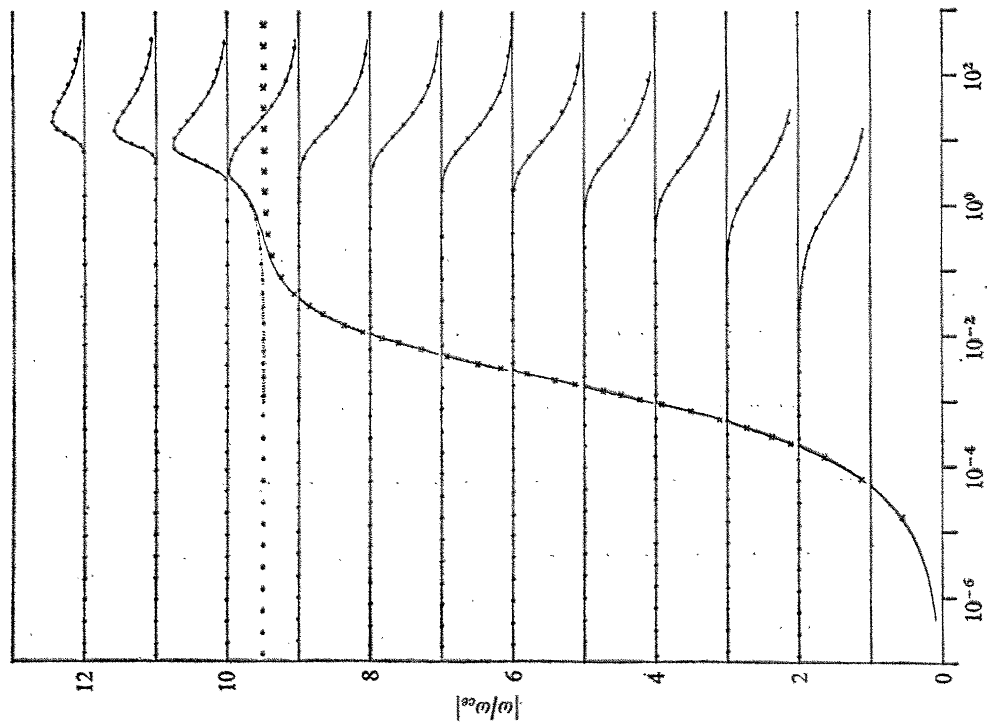

FIGURE 8. Generalized Bernstein modes.

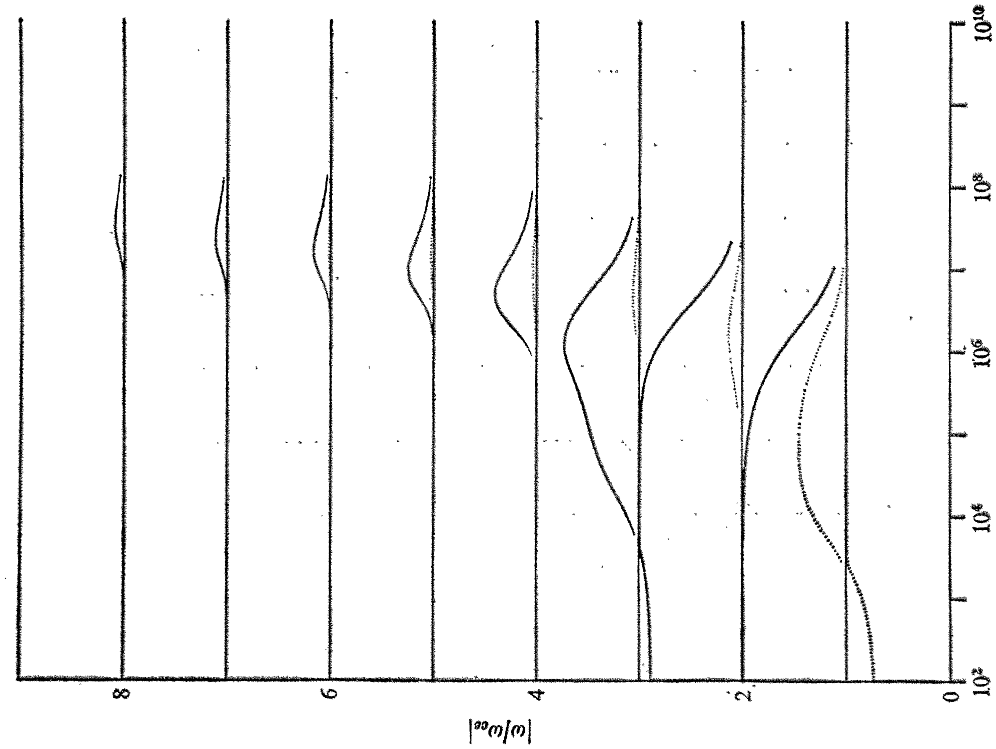

FIGURE 7. Generalized Bernstein modes.

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$\gamma = KT/mc^2$ is the logical choice. (It should be noted that *both* the two previous approximations imply $\gamma = 0$, but in one case it is KT, and in the other case $1/mc^2$, that are non-zero and finite.) Figures 5 and 6 correspond to $\gamma = 1.68 \times 10^{-3}$ and 1.68×10^{-2} respectively. Generally, the results combine the features of both figures 3 and 4 except in regions where the curves would cross or where one or other approximation would fail; near intersections of the curves they are modified and relinked. The 'upper hybrid resonance' region of cold plasma theory ($k_\perp \to \infty$) becomes a limited plateau, which connects with the branch of the Bernstein modes that originates at small $k_\perp$; this can be seen in figure 5. As γ increases, this plateau is reduced, figure 6. If $\omega_p/\Omega < \sqrt{3}$, so $\omega_H < 2\Omega$, the situation is somewhat different (figure 7) in the region $\omega < 2\Omega$, resulting in a larger 'plateau'.

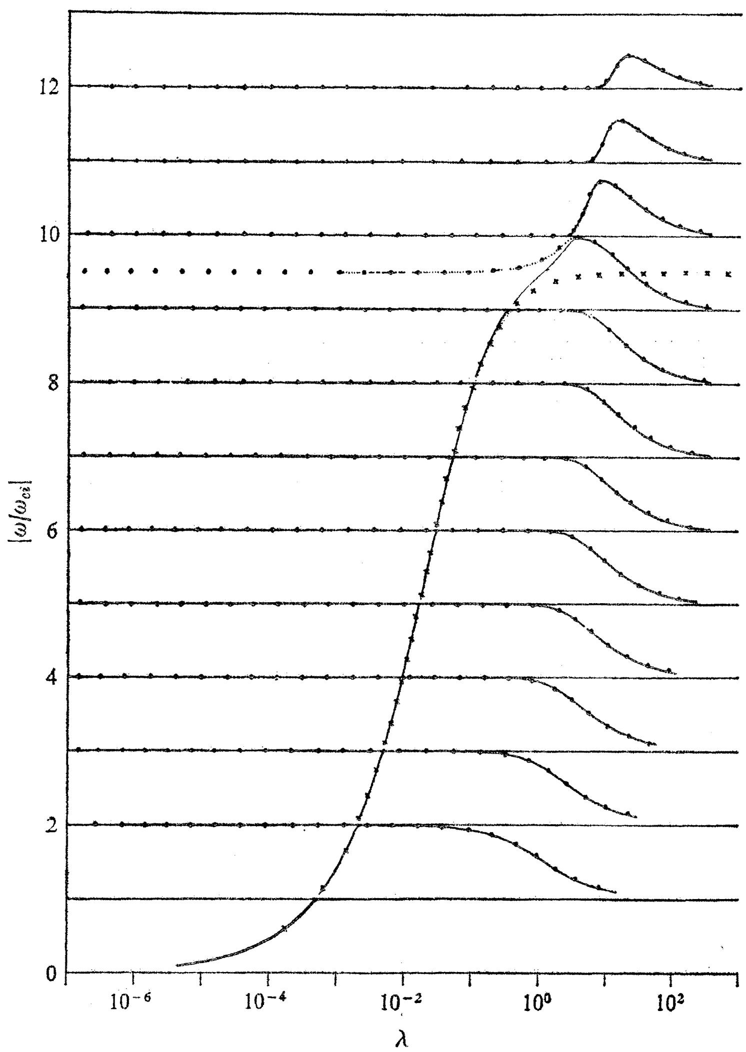

FIGURE 9. Generalized Bernstein modes.

Puri *et al.* (1973) also made similar calculations in respect of the ion modes. Here ω/Ω and λ refer to ion parameters, and the diagrams show the manner in which the Alfvén (now magnetosonic wave) is linked with ion Bernstein modes (figures 8 and 9). The lower hybrid resonance is again replaced by a plateau. It may be noted that for these calculations the mass ratio of ions to electrons was taken to be that of deuterium.

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Appendix

I was asked in the discussion to provide a *physical* explanation or description of waves in a hot plasma, and in particular the Bernstein modes. After some hesitation I offered the following, which readers may find helpful, though they should bear in mind that this kind of argument can be hazardous.

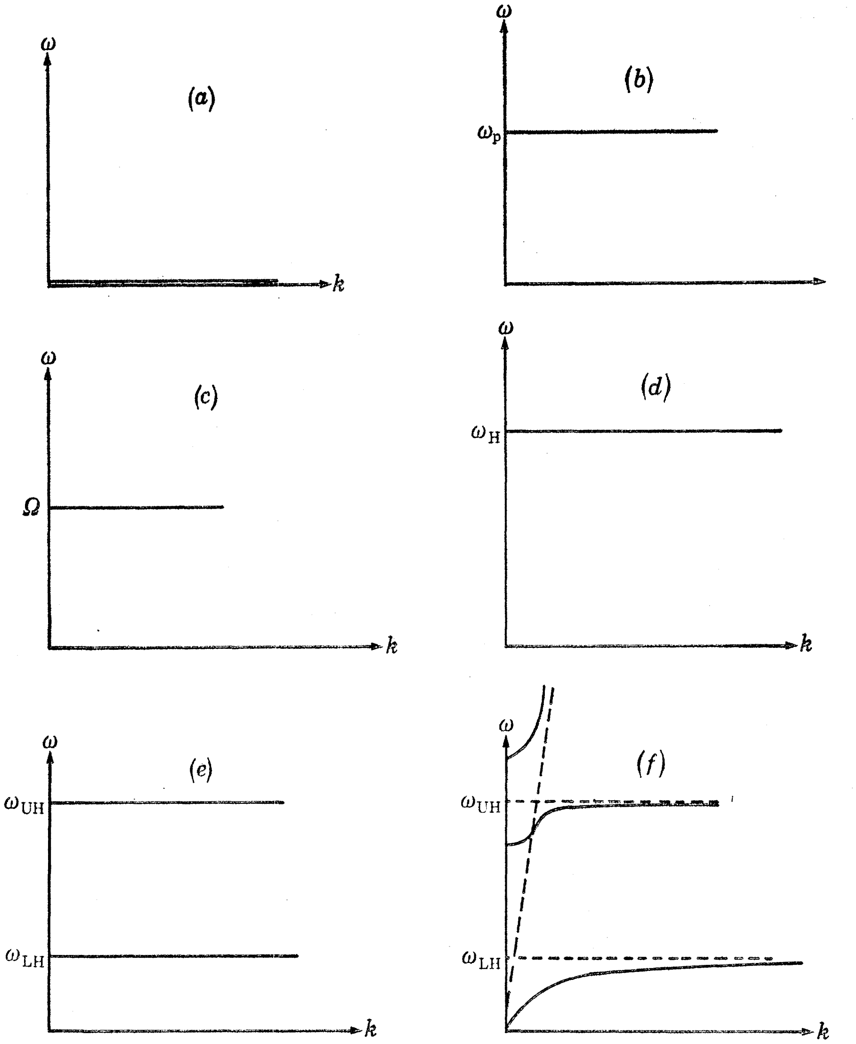

Figure 10. Development of dispersion relations: cold plasma.

The explanation is presented throughout in terms of the diagrams of ω plotted against k already used in the paper, and the idea is to consider the qualitative amendments required by the successive introduction of additional physical complications. We first illustrate this with the cold plasma (figure 10).

For a cold uniform gas with no physical interactions at all, an initially imposed disturbance of density merely persists without oscillating; thus $\omega(k) = 0$ for all k (figure 10*a*). If we give the particles of the gas *charge* (but with a fixed background of the opposite charge, in the usual way) we have an electrostatic restoring force, and we find plasma oscillations (figure 10*b*); increasing the density or the charge per particle, increases the plasma frequency ω_p. Electrostatic effects thus tend to push the dispersion curve upwards.

Now suppose there is a uniform magnetic field $\boldsymbol{B}_0$ and we fix attention on waves travelling perpendicular to $\boldsymbol{B}_0$. Suppose the $\boldsymbol{v} \times \boldsymbol{B}_0$ force must be taken into account but we can ignore mutual forces between the particles. If the particles are redistributed but left at rest the situation would again be as in figure 10*a*, but if the particles are allowed any slight motion they will gyrate in small circles with angular frequency Ω; the result is shown in figure 10*c*. If we now reintroduce the electrostatic interaction, the additional restoring force increases the frequency to the upper-hybrid $\omega_H = (\omega_p^2 + \Omega^2)^{\frac{1}{2}}$ and we have figure 10*d*. Next, allowing the positive ions to take part in the motion introduces a second branch in the dispersion curve as in figure 10*e*; this new branch has a similar physical description, and the frequency is the lower hybrid frequency. Finally we note that in figures 10*d* and *e* we assumed purely electrostatic interaction whereas a full theory must account for the finite speed of light, *c*. This modifies figure 10*e* to form figure 10*f*, and we note that figure 10*e* actually provides the asymptotic form of the curves in figure 10*f* for low values of ω/k, i.e. phase speed $\ll c$. Modification is necessary in a wedge close to the vertical axis, where the phase speed approaches or exceeds *c*. We note that (i) the branch at the upper hybrid frequency is abruptly deflected (on this scale) to terminate (at $k = 0$) at the cutoff frequency (this is the 'extraordinary' mode), (ii) a completely new branch appears, starting at the other cutoff frequency and lying wholly in the wedge; it corresponds asymptotically to vacuum propagation, (iii) the wave characteristic of the ions is modified close to the origin to become the Alfvén wave which in the limit of small *k* does not involve space change at all and can only be described by the full Maxwell equations. Figure 10*f* is of course just figure 4.

We emphasize that the introduction of a new physical ingredient in the theory can modify existing modes, introduce new modes, or both.

Now let us try a similar development for waves in a hot plasma. First, take the case of no external magnetic field. We begin with a gas of non-interacting particles. In the unperturbed uniform state each particle moves in a straight line at constant velocity. If we disturb all the particles by perturbations that are initially all sinusoidal in space, such perturbations are simply convected along. Grouping together those particles with the same value of $\boldsymbol{v}$, the convection of the perturbation simply results in a temporal oscillation at frequency $\boldsymbol{k} \cdot \boldsymbol{v}$. But a continuum of values of this frequency occur, and to the macroscopic observer (measuring, say, the total density) these all get superposed. If the perturbations were all in phase initially they will gradually get out of phase, so that macroscopically the perturbation appears to decay. Thus ω is purely imaginary. This is the essence of Landau damping. The details will depend on the population of each group and the extent of the perturbation applied to each group. Restoring the electrostatic interaction complicates this picture, and as we have seen increases (from zero) the real part of ω. For sufficiently small *k* (large wavelength) the Landau damping is vastly postponed, and ω becomes ω_p.

We turn now to the case where there is an external field $\boldsymbol{B}_0$ and propagation is to be perpendicular to it. The unperturbed motion of a particle is a spiral with axis parallel to $\boldsymbol{B}_0$, but we need only be concerned with the motion in the plane perpendicular to $\boldsymbol{B}_0$. Considering just one species (electrons) we have steady motion around a circle. As before we can group the particles according to their unperturbed velocity, and though each group is associated with a different radius, the angular velocity is the same, Ω, for all particles. The *centres* of these orbits are of course uniformly distributed. Now impose, at time $t = 0$, a sinusoidal perturbation proportional to $\exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x})$.

The perturbations will now be convected around the various circular paths, resulting in exponential functions of sines and cosines (this is where the Bessel functions come from). But after a time $2\pi/\Omega$, all perturbations are restored to their original state. So the development in time of any macroscopic quantity, such as density, is periodic, with period $2\pi/\Omega$, but in general not purely harmonic with frequency Ω, having some more complicated wave form. However, it can be Fourier analysed into contributions at frequencies Ω, 2Ω, 3Ω, ... The dispersion relation thus takes the form shown in figure 11*a*; this should be regarded as the generalization of figure 10*c*. The introduction of thermal motion has added the new modes $\omega = 2\Omega, 3\Omega, \ldots$ for all k; these are embryonic 'Bernstein modes'. By careful contrivance of the population of the groups and the extent of the perturbation for each group it would be possible, in principle, to arrange for just one harmonic to occur instead of the whole sequence. The resulting phenomenon could be called a 'wave' in the sense that it is a perturbation proceeding harmonically in space and time, but it should be noted that there is no 'restoring force' engendered by the wave itself (though such might be regarded by a physicist as an essential characteristic of a wave); the oscillation of the perturbation results from the circular unperturbed motion.

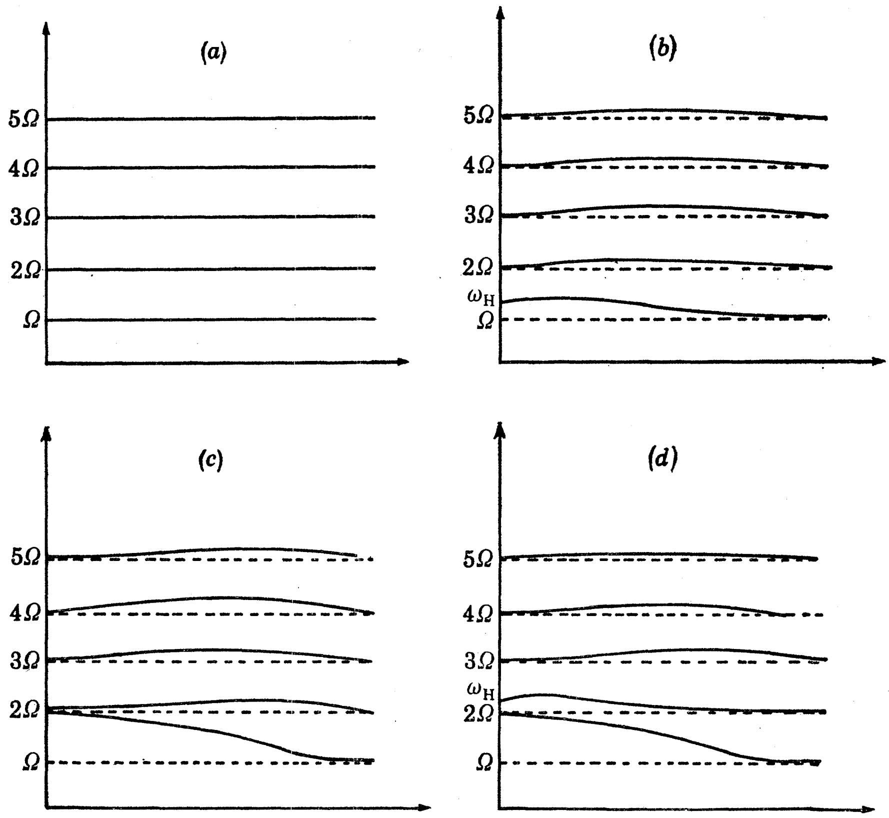

FIGURE 11. Development of dispersion relations: hot plasma.

Next we can reintroduce the electrostatic interaction between the particles; as previously its general effect is to push the dispersion curves upwards. We describe what happens as ω_p is slowly increased. The left hand portion of the lowest curve ($\omega = \Omega$) of figure 11*a* can be identified with the cold dispersion curve figure 10*c*, so it migrates upwards as in figure 10*d*, meeting the vertical axis at $\omega = \omega_H$; but for larger k the wave speed is slower and the effect of thermal modification becomes appreciable when $kR_L \sim 1$ where R_L is a typical Larmor radius. At large k the electrostatic interaction is unable to modify the curve, so it remains 'anchored' at $\omega = \Omega$ when $k \to \infty$, as shown in figure 11*b*. The curves $\omega = 2\Omega, 3\Omega, \ldots$ of figure 11*a* are also pushed upwards somewhat, especially around $kR_L \approx 1$, but are anchored at both $\omega = 0$ and $\omega = \infty$. It would appear that at these extremities of the curves the 'wave' is still essentially a matter of convection, since the Larmor radius of most of the particles is either much larger or much smaller than the wavelength.

As the plasma frequency is raised, ω_H increases until it reaches 2Ω (figure 11*c*). At this point a qualitative change in the curves must occur since the lowest one would have to cross the next one; in the event, for a Maxwellian plasma, the curves relink as in figure 11*d*. Continuing to raise ω_p, a further relinking occurs when ω_H reaches Ω, and so on. These features are readily seen in the computed curves given earlier in the paper.

When one introduces the possibility of ion motion, there arise 'ion Bernstein modes' whose physical description is similar. The diagram naturally becomes very complicated, but in practice, with $\Omega_i \ll \Omega_e$, the ion modes are only of interest for $\omega \ll \Omega_e$. The electrostatic approximation would then lead to a diagram with modes which at $k = 0$ have frequencies $2\Omega_i, 3\Omega_i, \ldots$, together with the lower hybrid frequency, and at $k \to \infty$ have frequencies $\Omega_i, 2\Omega_i, 3\Omega_i, \ldots$. But this needs correction near the origin owing to the failure of the electrostatic approximation there (Alfvén mode noted above).

The final step is to allow for finite light speed, as was done by Puri *et al.* (1973) (see § 4.3 above). Modification is necessary in the wedge $w > ck$, and near the origin, as already indicated in figure 10*f*. In those regions the curves are close to those of figure 10*f* itself, but where these would cross the curves of figure 11 there is relinking, as described in § 4.3.

Although the theory of Bernstein modes may seem abstruse, the electron modes are experimentally significant (topside resonances and laboratory experiments) and so are the ion modes (incoherent scatter); these topics have been extensively treated by other speakers at this meeting. It is hoped that the approach of this appendix will aid physical insight in this field.

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